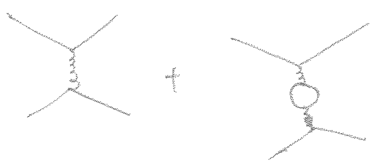
Jet lectures (1)

Thermal QCD

- o Literature & Bibliography
  - Kapusta & Gale: mostly Matsubara formalism
  - Le Bellac: more emphasis on real-time and thermal production rates
  - Laine lecture notes: concise exposition http://www.laine.unibe.ch/basics.pdf
- o At $T=0$: Reminder
  - QFT at $T=0$: small number of particles
  - primary observable is the scattering amplitude
  - $\langle \vec{p}_1 \vec{p}_2 \text{ out} | \vec{k}_1 \vec{k}_2 \text{ in} \rangle$, asymptotic "in" and "out" states at $t = \pm\infty$

LSZ reduction $\langle p_1 p_2 | k_1 k_2 \rangle \propto \langle vac | \overbrace{T\{\varphi(x_1)\varphi(x_2)\varphi(y_1)\varphi(y_2)\}}^{\hat{A}} | vac \rangle \Rightarrow$ vacuum to vacuum amplitudes

- Perturbatively:



$+ \dots \Rightarrow$ LOOP: vacuum fluctuation, Heisenberg uncertainty principle

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At finite T (finite density not treated here): large number of particles
- statistical techniques (statistical Field Theory) necessary, even classically
- initial condition NOT vacuum, but statistical ensemble: $\langle vac | \hat{A} | vac \rangle \Rightarrow \sum_i P_i \langle i | \hat{A} | i \rangle$

$= \mathrm{Tr}\big[\underbrace{\textstyle\sum_i |i\rangle P_i \langle i|}_{\hat{\rho} \text{ density operator}} \hat{A} \big]$ (initial state superposition of $|i\rangle$ with probs. $P_i \Rightarrow$ the state is not pure, but mixed)

$\Rightarrow$ Ensemble average $\langle \hat{A} \rangle = \dfrac{\mathrm{Tr}\, \hat{\rho} \hat{A}}{\mathrm{Tr}\, \hat{\rho}}$. In thermal eq. $\hat{\rho} = e^{-\beta(\hat{H} - \mu \hat{N})}$

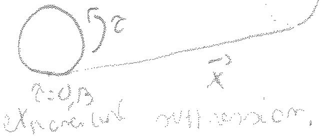
- What do we look at? No asymptotic states in the medium, no S-matrix less interesting
  - medium obs.
    - thermodynamics, EOS ($Z = \mathrm{Tr}\,\hat{\rho}$, $P = \frac{\partial}{\partial V} T \ln Z$)
    - transport properties ($\eta \dots$)
    - prod. rates of rare particles (γ, e^+e^-, …)
    - Jets …

- For static (time-independent) obs: Euclidean path integral

$$\frac{\mathrm{Tr}\, \hat{\rho} \hat{A}}{\mathrm{Tr}\, \hat{\rho}}, \quad \hat{\rho} = e^{-\beta H}$$

$\Rightarrow$ time evolution in the Euclidean direction ($e^{itH} \Leftrightarrow e^{-\beta H}$, $t \Leftrightarrow -i\beta$) with B.C. (arising from diagonal feature of the trace)

$$\Rightarrow \frac{\int D\varphi\, \hat{A}\, e^{-S_E}}{\int D\varphi\, e^{-S_E}} \qquad S_E = \int_0^\beta d\tau \int d^3x\, \mathcal{L}_E, \quad \varphi(0, \vec{x}) = \varphi(\beta, \vec{x})$$



$\Rightarrow$ well suited on the lattice

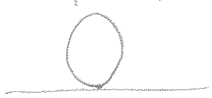
↳ Because exponential suppression, $\mu \neq 0 \Rightarrow$ sign problem

- Dynamical obs.
  - weak coupling, $\alpha_s \ll 1$
  - Analog models, AdS.

Jet lectures (2)

- In equilibrium, in the Matsubara formalism, B.C. cause a discretized frequency spectrum

$$\Delta_\varphi = \langle \varphi(p)\,\varphi(-p)\rangle = \frac{1}{\omega_m^2 + p^2} \qquad \omega_m = 2\pi m T,\ m\in\mathbb{Z}. \quad \text{For fermions } \pi(2m+1)T$$



$$\Rightarrow \quad T\sum_m \int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega_m^2+p^2} \underset{\text{exercise}}{=} \int\frac{d^3p}{(2\pi)^3}\Big(\underbrace{\frac{1}{2p}}_{\text{vacuum fluctuation}} + \overbrace{\frac{n_B(p)}{p}}^{\text{statistical / thermal fluctuation}}\Big) \qquad n_B(p) = \frac{1}{e^{\beta p}-1}$$

for quarks $\frac{1}{2} - n_F(p)$

$\Rightarrow$ at weak coupling, statistical distributions are $f(p) = \frac{1}{e^{\beta p} \pm 1}$

$\Rightarrow$ $g \ll 1$: gas of weakly interacting quarks and gluons

~~Assumes that~~ modes with $p \sim T$ dominate most of the thermo. observables (more on that later)

$\mathcal{E} \sim \int \underbrace{d^3p}_{T^3}\, \underbrace{p}_{T}\, \underbrace{f(p)}_{\sim 1} \sim T^4$, $\quad n \sim \int d^3p\, f(p) \sim T^3 \Rightarrow$ average separation $\Delta x \sim \frac{1}{n^3} \sim \frac{1}{T}$

for $p \sim T \Rightarrow$ wave packet in spread of $\lambda_c \sim 1/T$

$\Rightarrow$ hence when $p \sim T$ we are in the quantum regime

if $p \gg T$ $\quad \lambda_c \ll \Delta x \Rightarrow$ classical particles, Maxwell-Boltzmann gas

if $p \ll T$, bosons become classical fields $f(p) \gg 1$ (Rayleigh-Jeans regime)

~~sketch of the Bose distribution~~

$g \ll 1 \Rightarrow$ interactions.

When are interactions a perturbation and when are they not?

Typical fluctuation $\langle A^2\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\varepsilon_p}\big(1 \pm 2 f(p)\big)$, $\quad \langle A^2\rangle_T \equiv \int\frac{d^3p}{(2\pi)^3}\frac{f(p)}{\varepsilon_p} \sim T^2$ at the scale T

if interaction (fluctuation) $\ll$ kinetic term $\quad$ modes are perturbative

Def $\bar{A} = \sqrt{\langle A^2\rangle}$, covariant derivative $\partial_\mu + igA_\mu \sim p_\mu + ig\bar{A}$

Typical "hard" modes (dominate thermo) $\quad \underset{T}{p_\mu} + ig\underset{T}{\bar{A}} \Rightarrow$

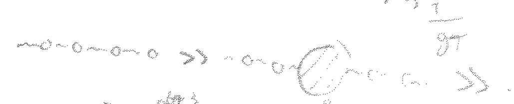
Soft modes $\quad \underset{gT}{p_\mu} + ig\underset{T}{\bar{A}} \Rightarrow$

Resummation (HTL, EQCD, more later)

$\Rightarrow$ for instance, Debye mass arises, $m^2 = g^2\int\frac{d^3p}{(2\pi)^3}\frac{f(p)}{\varepsilon(p)} \sim g^2T^2$

But: interactions among them (bosonic) $\langle A^2\rangle_{gT} = \int^{gT}\underbrace{\frac{d^3p}{(2\pi)^3}}_{g^3T^3}\frac{f(p)^{\to 1/g}}{\varepsilon(p)_{\to \frac{1}{gT}}} \sim gT^2$

$\Rightarrow \quad \underset{gT}{p^\mu} + ig\underset{gT}{\bar{A}_T} + ig\underset{g\sqrt{g}T}{\bar{A}_{gT}} \Rightarrow$



Ultrasoft modes $\quad$ Def : $p \sim g^2T$. What happens? $\langle A^2\rangle_{g^2T} = \int^{g^2T}\underbrace{\frac{d^3p}{(2\pi)^3}}_{g^6T^3}\frac{f(p)^{\to \frac{1}{g^2}}}{\varepsilon_p{}_{\to \frac{1}{g^2T}}} \sim g^2T^2$

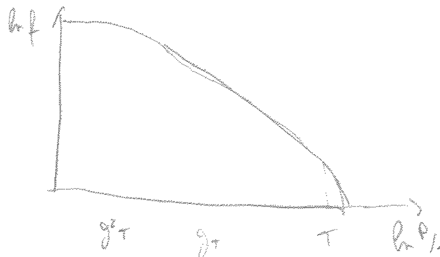
$\underset{g^2T}{p_\mu} + ig\underset{g^2T}{\bar{A}_{gT}} \Rightarrow$ interactions among US modes are non-pert.

MQCD Bödeker

Jet lectures (3)

Intermediate summary of scales (more later)

Bosonic modes



hard T, perturbative, dominates thermodynamics (moments)

soft gT, needs $O(1)$ resummations of hard modes, but self-interactions nonrelativistically perturbative

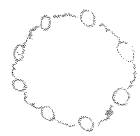
US g^2T, contributes no pert., no small parameter

Simple power-counting rules: when do these scales matter?

$\varepsilon, P \sim \int \frac{d^3p}{(2\pi)^3}\, p f(p)$

Hard: $T^4\left(1 \quad + \quad g^2 \quad + g^4 \quad + g^6 + \dots \right)$

Soft: $T^4\left(\quad g^3 + g^4 + g^5 + g^6 + \dots \right)$



$\underbrace{d^3p}_{\to g^3T^3}\ \underbrace{p}_{gT}\ \underbrace{f(p)}_{T/g} \Rightarrow g^3T^4$, $O(g)$ correction from $\langle A^2\rangle_{gT}$

Ultrasoft $T^4\left(\quad g^6 \right)$ $\quad \underbrace{d^3p}_{\to g^6T^3}\ \underbrace{p}_{g^2T}\ \underbrace{f(p)}_{T/g^2} \sim g^6T^4$

Some detail on the gT soft sector

- Field-theoretical approach: HTL, EFT, Braaten-Pisarski
- $\boxed{\text{Kinetic th. approach}}$ Wong, Blaizot-Iancu, ... QED example

$\frac{dN_a}{d^3p\, d^3\vec{x}} \equiv f_a(\vec{p},\vec{x},t)$ classical ph. space distribution ✓ Evolves without collisions, only under the influence of average $\vec{E}$ and $\vec{B}$ (weak spatial dependence)

↳ phase space

$\Rightarrow$ Vlasov equation $\quad \overbrace{\frac{\partial f_q}{\partial t} + \underbrace{\vec{v}}_{d\varepsilon_p/d\vec{p}} \cdot \vec{\nabla}_x f_q}^{v\cdot\partial_x} + \vec{F}\cdot \frac{\partial f_q}{\partial \vec{p}} = \overset{\text{collision term}}{0}$ (q: charge)

↳ $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$ Lorentz force

- $\vec{E}$ and $\vec{B}$ (average, long wavelength) sourced by (slow spatial variation of the plasma) $J^\mu_{ind} = e\int \frac{d^3p}{(2\pi)^3}\, \underbrace{v^\mu}_{(1,\vec{v})\equiv v^\mu} \left(f_+(\vec{p},\vec{x},t) - f_-(\vec{p},\vec{x},t) \right) \overset{\text{linear response}}{=} \int d^4y\, \Pi^R_{\mu\nu}(x-y) A^\nu($

in eq. $f_q(\vec{p},\vec{x},t) = f^0(\varepsilon_p)$, $J^\mu_{ind} = 0$. In linear approx. (small deviation from eq.) $f_q = f^0(\varepsilon_p) + \delta f_q(\vec{p},\vec{x},t)$

$\Rightarrow (v\cdot\partial_x)\,\delta f_q(\vec{p},\vec{x},t) = -\vec{F}\cdot\frac{\partial f^0}{\partial\vec{p}} = -\vec{F}\cdot\frac{\partial\varepsilon_p}{\partial\vec{p}}\frac{df^0}{d\varepsilon_p} = -\vec{F}\cdot\vec{v}\,\frac{\partial f_0}{\partial\varepsilon_p} = -q\,\vec{E}\cdot\vec{v}\,\frac{df^0}{d\varepsilon_p}$ [no magnetic contrib. f isotropic, $\frac{\partial f^0}{\partial\vec{p}} \parallel \vec{v}$

Def: $\delta f_q(\vec{p},\vec{x},t) \equiv -qW(\vec{x},\vec{v},t)\,\frac{df^0}{d\varepsilon_p}$, local perturbation of the momentum distribution of the particles

because $f(\vec{p},\vec{x},t) = f^0(\varepsilon_p) - qW(\vec{x},\vec{v},t)\frac{df^0}{d\varepsilon_p} \approx f^0(\varepsilon_p - qW(\vec{x},\vec{v},t))$

Eq. for W: $\quad v\cdot\partial_x\,\delta f_q = v\cdot\partial_x\left[-q\,W(\vec{x},\vec{v},t)\,\underbrace{\frac{df^0}{d\varepsilon_p}}_{x\text{-indep.}}\right] = -q\,\vec{v}\cdot\vec{E}\,\frac{df^0}{d\varepsilon_p} \Rightarrow v\cdot\partial_x W(\vec{x},\vec{v},t) = \vec{v}\cdot\vec{E}$

Jel lectures (4).

Can solve using characteristics. $v\cdot\partial_x$: line derivative along the characteristic defined by $\frac{d\vec{x}}{dt}=\vec{v}$

$$\Rightarrow \begin{cases} \frac{dW}{dt} = v\cdot E \\ \frac{d\vec{x}}{dt} = \vec{v} \\ \frac{d\vec{v}}{dt} = 0 \quad (\text{No force}) \end{cases}$$

retarded b.c., perturbation vanishing at $t=-\infty$

$$\Rightarrow W(\vec{x},\vec{v},t) = \int_{-\infty}^{t} dt'\, e^{-\hat{\varepsilon}(t-t')}\, \vec{v}\cdot\vec{E}\left(\vec{x}-\vec{v}(t-t'),t'\right)$$

or schematically $W(\vec{x},\vec{v},t) = \frac{1}{v\cdot\partial_x}\,\vec{v}\cdot\vec{E}$

$$J^{\mu}_{ind}(E) = -2e^2\int\frac{d^3p}{(2\pi)^3}\, v^{\mu}\,\frac{df^0}{d\varepsilon_p}\int_0^{\infty}d\tau\; e^{-\varepsilon\tau}\;\vec{v}\cdot\vec{E}\left(\vec{x}-\vec{v}\tau,\,t-\tau\right)$$

we are working with a small perturbation, linear response theory $\Rightarrow J^{\mu}_{ind}(\vec{x},t) = \int d^4y\; \Pi^{R}_{\mu\nu}(x-y)A^{\nu}(y)$

Using $\vec{E} = -\partial_0\vec{A}-\vec{\nabla}A_0$ and $\int d^4x\, e^{iQ\cdot x}\int_0^{\infty}d\tau\, e^{-\varepsilon\tau} f(x-v\tau) = \frac{i f(Q)}{v\cdot Q+i\varepsilon}$ and ASSUMING MASSLESS PARTICLES, $|\vec{v}|=1$

Exercise

$$\Rightarrow \Pi_{\mu\nu}(\omega,q) = m_D^2\left\{-\delta_{\mu 0}\delta_{\nu 0} + \omega\int\frac{d\Omega_v}{(4\pi)}\,\frac{v_\mu v_\nu}{\omega-\vec{v}\cdot\vec{q}+i\varepsilon}\right\},\quad m_D^2 = -\frac{2e^2}{\pi^2}\int_0^{\infty}dp\,p^2\,\frac{df^0}{dp} = \frac{4e^2}{\pi^2}\int_0^{\infty}dp\,p f$$

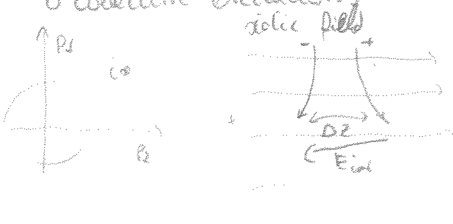
QCD: $(v\cdot D_x)^{ab}W_b(\vec{x},\vec{v},t) = \vec{v}\cdot\vec{E}^a(\vec{x},t)$

QCD $= \frac{2g^2}{\pi^2}\int_0^{\infty}dp\left(N_c\, n_B(p) + N_f\, n_F(p)\right)$

Exercise: diagrammatic calculation

Intuitive picture of some consequences

o collective excitations

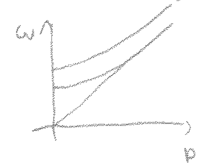


$\Rightarrow \vec{F}=g\vec{E}$ causes a separation $\vec{F} = \frac{d\vec{p}}{dt} = \varepsilon(p)\frac{d\vec{v}}{dt} \Rightarrow \Delta z \sim \frac{gEt^2}{\varepsilon(p)}$

$$\text{induced } \vec{E}_{ind} \sim \int \underbrace{g\,\Delta z}_{\text{for one part.}}\;\underbrace{f(p)\,d^3p}_{\text{number of part.}} \sim g^2\int\frac{d^3p}{\varepsilon(p)}\, f(p)\,\vec{E}\,t^2 \sim m_D^2\, E t^2$$

if $t\sim\frac{1}{m_D}$ $\vec{E}+\vec{E}_{ind}\sim 0 \Rightarrow$ no $\vec{E}$ field, particles fly apart until attraction kicks in $\Rightarrow$ oscillation

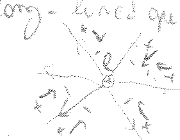
$E \propto \sin(\omega_p t)$ plasmon frequency, static electric fields not allowed!
$\hookrightarrow \omega_p \sim gT$ $\boxed{\text{Plasma oscillations}}$



plasmon dispersion relation

- lifetime of these excitations is $O(g^2T) \Rightarrow$ long-lived quasi particles

o screening similar phenomenon



A^0 static, slowly varying over space ($\partial^0 A^0 = 0$)
phase-space distribution affected as

$$\delta n(\vec{x},\vec{p},t) = \frac{1}{e^{(p+eA^0)/T}\pm 1} - \frac{1}{e^{p/T}\pm 1} \simeq -e\,A^0(\vec{x},t)\,\frac{n(p)(1\pm n(p))}{T} \Rightarrow J^0_{ind} = e\int\frac{d^3p}{(2\pi)^3}\, v^0\,\delta n = -m_D^2 A^0(\vec{x},t)$$

$\hookrightarrow$ induced charge density

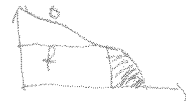
o Landau damping $\frac{1}{\omega-\vec{v}\cdot\vec{q}+i\varepsilon} = P\frac{1}{\omega-\vec{v}\cdot\vec{q}} - i\pi\,\delta(\omega-\vec{v}\cdot\vec{q})$

Polarization has imaginary part for SPACELIKE modes ($\omega=\vec{v}\cdot\vec{q}$, $|v|=1$)

$\Rightarrow$ (chromo)e.m field transfers energy to plasma particles whose velocity along q ($v_p = \hat{q}\cdot\vec{v}$) is equal to the phase velocity of the e.m field $\left(\frac{\omega}{q}\right)$
(reminder: phase velocity $\frac{\omega}{k}$, group velocity $\frac{d\omega}{dk}$, different in a medium)

Jet lectures (5)

Dynamical scales, transports and jets



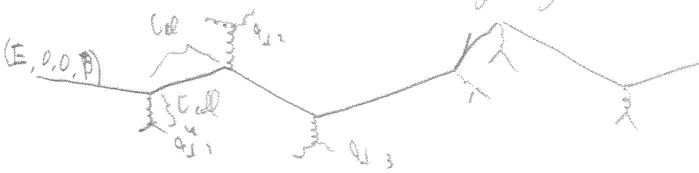
$\Rightarrow$ hard $p \gtrsim T$ modes

$1/T$: average separation between hard modes

$\frac{1}{gT}$: collective excitations, (chromo)electric screening, Landau damping

$1/g^2T$: magnetostatic screening (no colored MHD)

Now let us try to study a propagating hard mode, $P_0^2 - \vec{P}^2 \lesssim g^2T^2$ (on shell), $P^0, P \gtrsim T$



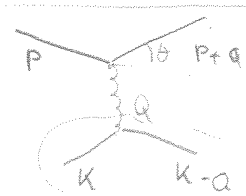
Elastic collisions are the most effective in deflecting $\Rightarrow$ momentum broadening gained $Q_\perp^2 = q_{\perp 1}^2 + q_{\perp 2}^2 + q_{\perp 3}^2 + \dots$

(*) if the duration of a collision is $\ll$ time between collisions $\Rightarrow$ uncorrelated kicks, random walk in momentum

$$|Q_\perp| \propto \sqrt{N_{coll}} \propto \sqrt{t} \quad \Rightarrow |Q_\perp|^2 = \hat{q}\, t \qquad \hat{q} \sim \frac{\Delta q_\perp^2}{l_{el}}$$

$\Rightarrow$ elapsed time

(*) Does (*) apply? Soft scatterings



$\theta \sim q_\perp/p$

thermal constituent of the plasma

$$\frac{d\Gamma}{d^2q_\perp} \sim \int dq_z \int d^3k\; f(k)\,(1 \pm f(k+q))\; \frac{d\sigma}{d^3q}$$

if q soft, $q \sim gT \ll K$ $\quad \sim \underbrace{\int d^3k\, f(k)\,(1 \pm f(k))}_{T^3} \frac{d\sigma}{d^2q_\perp}$

$$\frac{d\sigma}{d^2q_\perp} \sim \frac{g^4}{(q_\perp^2)^2} \quad \Rightarrow \quad \Gamma \sim g^4T^3 \int d^2q_\perp \frac{1}{q_\perp^4} \sim g^2T \Rightarrow l_{el} \sim \frac{1}{g^2T},\; t_{coll} \sim \frac{1}{q_\perp} \sim \frac{1}{gT} \Rightarrow \text{works}$$

Warning: very rough estimate. Proper treatment of the medium, electric screening. $\frac{1}{(q_\perp^2)^2} \to \frac{1}{q_\perp^2(q_\perp^2 + m_D^2)}$

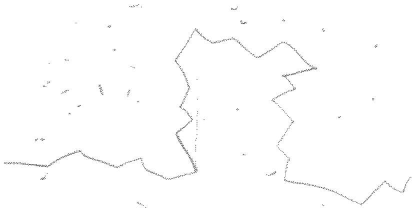
$\Rightarrow$ total rate still divergent, but $\hat{q}$ now IR finite

$$\boxed{\hat{q}_{gT} \sim \int^{gT} d^2q_\perp \frac{q_\perp^2}{q_\perp^2(q_\perp^2+m_D^2)^2}\, g^4T^3 \sim g^4T^3}$$

$\Rightarrow$ logarithmically insensitive

Large-angle scattering, $q_\perp \sim T$, $t_{coll} \sim 1/T$, $l_{large} \sim \frac{1}{g^4T}$ $\Rightarrow$ much longer mean-free path,

but equally important for $\hat{q}$ $\Rightarrow$ $\hat{q}_T \int^T d^2q_\perp \frac{q_\perp^2}{q_\perp^4}\, g^4T^3 \sim g^4T^3$



$\Rightarrow$ large angle scatterings important for determining transport properties (η for instance) $\Rightarrow$ Boltzmann regime

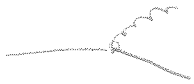
$\Rightarrow$ on timescales longer than $1/g^4T$ $\Rightarrow$ Hydro description

Hydro arises also in strongly coupled systems without scale separations and quasi-particle picture

$\Rightarrow$ WEAKLY COUPLED picture is a sufficient condition

Jet lectures (6)

- Radiative processes also ~~happen~~ on a $\frac{1}{g^4 T}$ timescale:

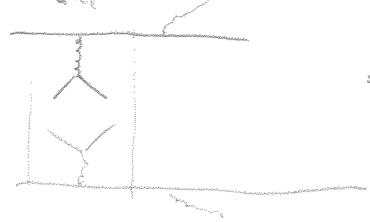


$\Rightarrow$ not possible, no phase space

- Need a collision to kick off-shell and open phase space
  - most common collisions soft

$\Rightarrow$ Naive Bethe-Heitler (Gunion-Bertsch)

$d\Gamma_{el}$



$\Rightarrow \Gamma_{split} \sim g^2 \Gamma_{el} \sim g^4 T$

$\Rightarrow$ same as large angle scattering, important for transport

- Long formation time introduces interference, LPM suppression

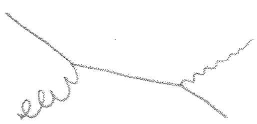
$\Longrightarrow$ See E. Iancu's lectures

$\boxed{\text{Thermal production rates}}$

- Well posed question to ask: how often is a $\underline{\text{rare}}$ particle thermally produced? (particle not part of heat bath / plasma d.o.f.)
  - relevance for cosmology too
- basic picture: probe (particle) weakly coupled to the plasma through $J\varphi$ $\to$ rare probe
  - example: photon $J^\mu = e \sum_f \bar\psi_f \gamma^\mu \psi_f$, A_μ e.m. field

$$\Rightarrow \frac{d\Gamma}{d^3k} = -\frac{1}{(2\pi)^3 2k^0} \underbrace{\int d^4X \, e^{ik^0 x^0 - i\vec{k}\cdot\vec{x}}}_{\substack{\text{Minkowskian, real photon} \\ k^0 = |\vec{k}| \sim T}} \underbrace{\langle J^\mu(0) J_\mu(X) \rangle}_{\Pi^<(X),\ \text{th. average of } JJ \text{ correlator}}$$

in p.t., at LO

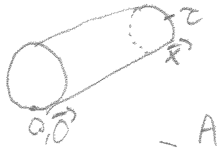


$\Rightarrow$ large-angle scattering processes, $g^4 T \Rightarrow g^2 e^2 T (T^3)$

$d\Gamma_{el}$

$\Rightarrow g^2 e^2 T (T^3)$

- On the lattice? Can measure $\Pi_E(\tau, \vec{k}) = \int d^3x \, J_\mu(\tau, \vec{x}) J_\mu(0,0) e^{i\vec{k}\cdot\vec{x}}$



- Analytical continuation prescription: $\Pi_E(\tau, \vec{k}) = \Pi^<(i\tau, \vec{k})$, $0 < \tau < 1/T$
- At equilibrium $\Pi^<(k^0, \vec{k}) = n_B(k^0) \rho(k^0, \vec{k})$

ρ spectral function of the Euclidean current

$$\Pi^<(t, \vec{k}) = \int_{-\infty}^{+\infty} \frac{dk^0}{(2\pi)} e^{-ik^0 t} n_B(k^0) \rho(k^0, \vec{k})$$

$$\Rightarrow \Pi_E(\tau, \vec{k}) = \Pi^<(i\tau, \vec{k}) = \int_{-\infty}^{+\infty} \frac{dk^0}{(2\pi)} e^{k^0 \tau} n_B(k^0) \rho(k^0, \vec{k}) \overset{\text{symmetries}}{=} \int_0^\infty \frac{dk^0}{(2\pi)} \rho(k^0, \vec{k}) \frac{\cosh k^0(\tau - \frac{1}{2T})}{\sinh \frac{k^0}{2T}}$$